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Geometrical Optics as an Abelian U(1) Gauge Theory in a Vacuum Space-time

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Abstract

As a consequence that geometrical optics (the eikonal equation) can be derived from Maxwell's equations and Maxwell's theory is nothing but an Abelian U(1) gauge theory, we propose that geometrical optics could also be treated as an Abelian U(1) gauge theory. We formulate geometrical optics as an Abelian U(1) gauge theory in a (3+1)-dimensional vacuum space-time as an approximation of the weak field. We show the explicit form of the phase, the gauge potential, and the field strength tensor related to the refractive index. We calculate numerically the refractive index and the magnetic field using the suitable parameters that we choose to mimic the real condition of nature. We obtain (without unit) the values of the refractive index $n(\vec{r})$=1.0001 to represent a vacuum space-time and the amplitude $\rho = 0.55853$ related to magnitude of the magnetic field $|\vec{B}| = 0.10452$ to represent the weak field. The view of geometrical optics as gauge theory could be generalized or related to topological field theory where geometrical optics could have a topological structure in the caseof the weak field.

Introduction

Why geometrical optics? Geometrical optics is a simplified theory than the electromagnetic wave of Maxwell's theory to describe the behaviour of light. The geometrical optics corresponds to the limiting case of a very small wavelength of light, $\lambda \to 0$ [1,2], in comparison with the characteristic dimension of the problem [3] i.e. the system with which light interacts [4] or to each of the other scales present (the scale of observation [5]), so that theelectromagnetic waves can be regarded locally as plane waves propagating through space-time [6]. Geometrical optics employs the concept of rays, which are defined as the direction of propagation of energy in the limit $\lambda \to 0$ [4]. The fact that $\lambda \to 0$ in comparison with the scale of observation is called the classical limit [7].

As an example that geometrical optics is a simpler theory than electromagnetic wave, we can observe that the spreading of light due to diffraction is entirely because of the finiteness of the wavelength. However, if we assume that the wavelength tends to zero then we can perform the infinitesimally thin beam of light that defines rays [4].

It is commonly considered that an Abelian $U(1)$ local gauge theory describes Maxwell's theory. It is very rare to find an article that discusses geometrical optics as an Abelian $U(1)$ local gauge theory [8,9]. Geometrical optics as an Abelian $U(1)$ local gauge theory in a (3+1)-dimensional vacuum space-time had been considered [9]. There [9] the electromagnetic wave or light ray propagation had been formulated as the dispersion relation from which the eikonal-type equation can be reconstructed. They [8,9] do not formulate the explicit form of the phase related to the refractive index. Also, it [9] says nothing that a vacuum could be approximated using the weak-field limit.

In this article, we propose geometrical optics as an Abelian $U(1)$ local gauge theory in a (3+1)-dimensional vacuum space-time as an approximation of the weak-field limit. We will formulate the explicit form of the phase (in turn the gauge potential and the field strength tensor) related to the refractive index and we will apply the idea of a vacuum as the weak-field limit to the geometrical optics. We consider these two points to be new ones.

Why could geometrical optics be treated as an Abelian $U(1)$ local gauge theory? One of the reasons is the geometrical optics (the eikonal equation) can be derived from Maxwell's equations [4,10,11]. Rays in geometrical optics are invariant under a local gauge transformation [12]. This view of geometrical optics as a gauge theory, in turn, could be generalized or related to topological field theory where geometrical optics could have a topological structure.

Why could geometrical optics in a vacuum space-time be treated as a weak-field limit theory? What we mean by a vacuum space-time is space-time where the field is weak. The field here can be an electric field, magnetic field, or electromagnetic field. This vacuum (weak-field) space-time is related to the space-time of the infinite radius, r, from sources (a charge, current). We could say that a vacuum (weak-field) space-time implies the isotropic space-time. A vacuum space-time which could be considered as the weak-field limit of the electromagnetic fields (consisting of a scalar field) had been proposed successfully a long time ago [13].

Abelian $U(1)$ Gauge Theory

Maxwell's theory is well known as an Abelian $U(1)$ local gauge theory. The treatment of the geometrical optics as an Abelian $U(1)$ local gauge theory implies the gauge potential of the geometrical optics and Maxwell's theory are the same, i.e. both are the Abelian $U(1)$ gauge potential which commonly can be written as

$$\vec{A}_\mu = \vec{a}_\mu(\vec{r},t)e^{iq(\vec{r},t)} \tag{1}$$

where $\vec{A}_\mu$ is a complex [6,14] gauge potential, $\vec{a}_\mu(\vec{r},t)$ is a complex amplitude [13], a slowly varying function of space coordinates and time [3], $q(\vec{r},t)$ is the eikonal (a real phase [6]), a function of space coordinates and time, $q(\vec{r},t)$ is a complex scalar function. $\vec{A}_\mu$ as a complex amplitude can be interpreted as the oscillating variable [15], the displacement from an equilibrium [16] i.e. a position at infinity where the gauge potential can be assumed equal to zero.

As an Abelian $U(1)$ local gauge theory, the field strength tensor of Maxwell's theory can be written as

$$\vec{F}_{\mu\upsilon} = \partial_\mu \vec{A}_\upsilon - \partial_\upsilon \vec{A}_\mu \tag{2}$$

where ∂_μ denotes the partial derivative (four-gradient) and μ denotes the three-dimensional space plus one-dimensional time, $\vec{A}_\upsilon$ is the gauge potential. It means that we have also such field strength tensor and related gauge potential in geometrical optics.

Electromagnetic, Subset and Weak Fields

Let us assume that the electromagnetic fields (the set of the solutions of Maxwell equations) in a vacuum space-time have a subset field [17], $\varphi(\vec{r},t)$ a scalar function of a space-time. Any electromagnetic field is locally equal to a subset field i.e. any electromagnetic field can be obtained by patching together subset fields (except in a zero-measure set) but globally different [17]. This means that the difference between the set of the subset fields and all the electromagnetic fields in Maxwell's theory ina vacuum space-time is global instead of local since the subset fields obey the topological quantum condition [17].

The electromagnetic field satisfies a linear field equation, but a subset field satisfies a non-linear field equation. Both fields, the electromagnetic field, and a subset field, satisfy the linear field equation in the case of the weak field [13]. In other words, the vacuum Maxwell's theory is the weak field limit [13] of a non-linear subset field theory. It means that a non-linear subset field theory reduces to the vacuum Maxwell's linear theory in the case of the weak field. A space-time where the weak field lives approximately represents a vacuum space-time.

As a consequence that the electromagnetic fields in a vacuum space-time have a subset field, a scalar function of space-time with all of its properties, we also have such subset field in geometrical optics which can be written as

$$\varphi(\vec{r},t) = \rho(\vec{r},t)e^{iq(\vec{r},t)} \tag{3}$$

and

$$f(\vec{r},t) = -1/[2\pi(1+\rho^2)] \tag{4}$$

where $\rho(\vec{r},t)$ is the amplitude, a slowly varying function of the coordinates and time [3] (or constant [18]), $q(\vec{r},t)$ is the phase, $f(\vec{r},t)$ is the function of amplitude. What we mentioned previously by the weak-field is $|\varphi\varphi^*| \ll 1$ where φ^* is the complex conjugate of φ. The assumptions that the set of the solutions of Maxwell equations in a vacuum space-time has a subset field has far-reaching consequences.

We call the functions, $f(\vec{r},t)$ and $q(\vec{r},t)$, shown in eqs. (3), (4), as the Clebsch variables [19] or Gaussian potentials [20]. These Clebsch variables are related to any divergence-less vector field [17]. An example of a divergence-less vector field is the magnetic field $\vec{B}$, where $\vec{\nabla} \bullet \vec{B} = 0$. The Clebsch variables are not uniquely defined. However, many different choices are possible for them [17]. The treatment (3) is based on the wave point of view of the field. We could interpret the scalar field as the disturbance where the physical disturbance is the real part ofa scalar field [21].

By using the Clebsch variables, f and q [19], the field strength tensor of the geometrical optics, eq.(2), can be written as [19]

$$\vec{F}_{\mu\upsilon} = \partial_\mu(f\,\partial_\nu q) - \partial_\nu(f\,\partial_\mu q). \tag{5}$$

The eqs. (2), (5) are the linear field equations.

By observing the equality of eqs. (2) and (5), we see that [19]

$$\vec{A}_\upsilon = f\,\partial_\nu\, q. \tag{6}$$

Eq. (6) shows that the gauge (vector) potential can bewritten using the Clebsch (scalar) variables.

Phase in Geometrical Optics

What is a phase, $q(\vec{r},t)$, in geometrical optics? Why phase? Let us introduce $\psi_1(\vec{r})$ which is called eikonal [3]. The relation between $\psi_1(\vec{r})$ and $q(\vec{r},t)$ can be expressed as [3]

$$\psi_1(\vec{r}) = \frac{c}{f_\theta} q(\vec{r},t) + ct \tag{7}$$

where $\psi_1(\vec{r})$ as we see explicitly is a function of space coordinates only [3], "a length" of line in space, a real [22] scalar function.

We can formulate phase in geometrical optics by substituting the eikonal equation, $|\vec{\nabla}\psi_1(\vec{r})| = n(\vec{r})$ [3], into eq.(7), we obtain [23]

$$q(\vec{r},t) = X(\psi_1 - ct) = X\left(\int_{r_1}^{r_2} n(\vec{r}) d^3r - ct\right) \tag{8}$$

where $X = f_\theta / c_1$ and $n(\vec{r})$ is the refractive index. The refractive index is the real scalar function of coordinates (vector position) with positive values, the slowness at a point [24]. The refractive index is typically supplied as known input, given, and we seek the solution, the phase [24]. The integral $\int_{r_1}^{r_2} d^3r$ shows the propagation of ray from the initial position, r_1, to the final position, r_2, in 3-dimensional space.

The importance of the phase is the value of the gauge potential, the field strength tensor, and in turn the energy is unchanged although we change the phase i.e. the phase or the gauge transformation. It differs from the other component of the scalar field, i.e. the amplitude. The energy of the physical system we observe is changing if we change the amplitude.

We see from eq.(9) that the phase is related to the refractive index. We could say that it is a new formulationof a Clebsch variable of the phase related to the refractive index. This phase related to the refractive index satisfiesthe divergence-less property of any vector field. The divergence-less property of any vector field means that there isno source to give rise to such a vector field. However, we could approximate "there is no source" with "the weak field related to the infinite radius from the source".

The infinite radius from the source implies the isotropicspace-time where the direction does not matter. In the isotropic space-time, the value of the refractive index is determined solely by the distance from the origin. That is why the property of a scalar field is isotropic (well-defined) for an infinite distance (radius) from the originor the source. The property of a scalar field as a function of space-time (physics) seems likely in harmony with the property of space-time (geometry). A space-time could be locally anisotropic but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

In the case of the time-independent, eq.(8) becomes

$$q(\vec{r}) = X \int_{r_1}^{r_2} n(\vec{r}) d^3 r. \tag{9}$$

Eq.(9) tells us that the phase transformation reduces to the refractive index transformation determined solely by the value of the distance from the origin or the source. The refractive index is a number and the phase is anangle in radians or degrees.

By substituting eq.(8) into eqs. (6), (5), the gauge po tential (6) and the field strength tensor (5) becomes

$$\vec{A}_{\upsilon} = f\, \partial_{\nu} \left[X \left(\int_{r_1}^{r_2} n(\vec{r}) d^3 r - ct \right) \right] \tag{10}$$

and

$$\vec{F}_{\mu\upsilon} = \partial_{\mu} \left\{ f\ \ \partial\nu \left[X \left(\int_{r_1}^{r_2} n(\vec{r}) d^3 r - ct \right) \right] \right\} - \partial_{\nu} \left\{ f \partial \mu \left[X \left(\int_{r_1}^{r_2} n(\vec{r}) d^3 r - ct \right) \right] \right\} \tag{11}$$

respectively. Eqs. (10), (11) are the explicit forms of the gauge potential and the field strength tensor formulationsin the geometrical optics, respectively.

Numerical Simulation

In the calculation of the refractive index where therefractive index decreases radially out from the origin(the location of sources), we use the relation below [25]

$$n(\vec{r}) = n_0 \left(1 - \frac{ar^2}{2} \right) \tag{12}$$

where n_0 is the maximum refractive index, r is the distance, a is the positive (the gradient) constant, a function of wavelength and it depends on the specific gradient index (GRIN) material [25].

In order to mimic the condition of the real world, we choose the value of parameters as follows: n_0 = 1.6, a =0.7499, r = $|\vec{r}|$ = 1. With the help of AI and Octave, we calculate numerically the refractive index (12) and obtainthe result $n(\vec{r})$ = 1.0001. This result indicates a vacuum space-time. Variation of refractive index with distance is shown in **Fig. 1.**

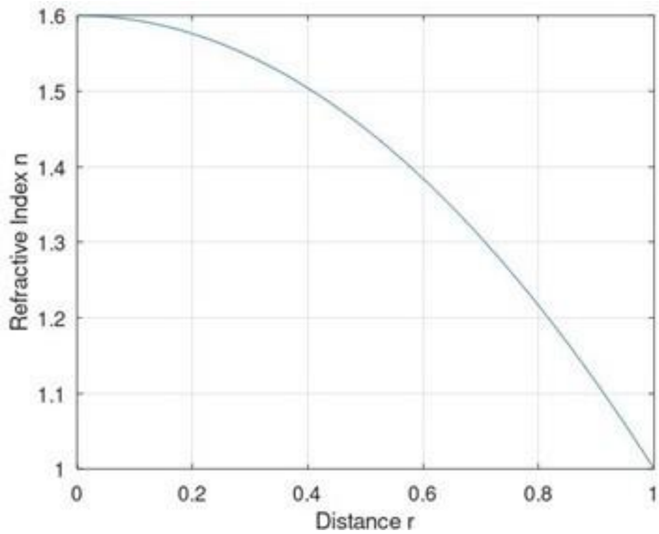

Fig. 1 Variation of refractive index with distance.

As we see from **Fig. 1**, it shows us the value of therefractive index decreases (increases) when the distance increases (decreases).

To simplify the complexity of the numerical computation of the field strength tensor, we will compute numerically one of the components of the field strength tensor i.e. the the magnetic field, $\vec{B} = \vec{\nabla} \times \vec{A}$, where $\vec{A}$ is the magnetic (vector) potential (the other component is the electric field). The choice of the magnetic field accommodates the divergence-less vector field as we mentioned previously where the Clebsch variables are related.

We choose the value of parameters as follow: $f_\theta = 299792458, t = 1, \rho = linspace(-5,5,1000)$. We take the value of the speed of light in a vacuum c = 299792458.We obtain the relation of ρ versus $\vec{B}$ as shown in **Fig. 2** below. As an example, take a maxima peak point of the amplitude $\rho = 0.55853$, it is related to magnitude of the magnetic field $|\vec{B}| = 0.10452$.

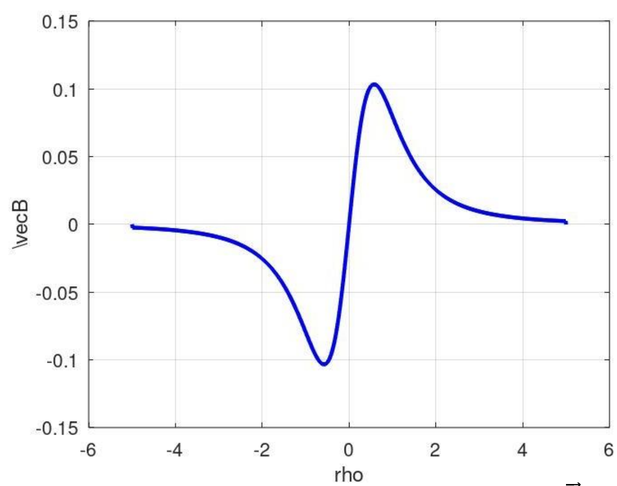

Fig. 2 The graph of ρ versus $\vec{B}$

We consider that ρ = 0.55853 is related to the weak field because $\varphi\varphi^*$ = ρ^2 = 0.55853^2 = 0.3119 << 1. This is the reason why the weak field can be interpreted as a vacuum space-time.

Discussion and Conclusion

The treatment of geometrical optics as an Abelian U (1) local gauge theory has consequences in that we have theconcepts of the phase, the gauge potential, and the field strength tensor in geometrical optics as in Maxwell's theory. The formulations of the phase, the gauge potential, and the field strength tensor in geometrical optics are related to the refractive index.

It is natural to assume that a vacuum space-time isthe weak-field limit. This vacuum (weak-field) space-time is related to the space-time of the infinite radius from sources (a charge, current). We could say that a vacuum (weak-field) space-time implies the isotropic space-time. We show numerically that a vacuum space-time and the weak field are indicated by the values of the refractive index, $n(\vec{r}) = 1.0001$, and the amplitude, ρ^2 = 0.3119 << 1, respectively. We choose the parameters in order to give the results that mimic the real condition in nature.

We see from eq.(8) that the phase is related to the refractive index. We could say that it is a new formulationof a Clebsch variable of the phase related to the refractive index. This phase related to the refractive index satisfiesthe divergence-less property of any vector field. The divergence-less property of any vector field means that there is no source to give rise to such a vector field. However, we could approximate "there is no source" with "the weak field related to the infinite radius from the source".

The infinite radius from the source implies the isotropicspace-time where the direction does not matter. In the isotropic space-time, the value of the refractive index is determined solely by the distance from the origin. Thatis why the property of a scalar field is isotropic (well-defined) for an infinite distance (radius) from the originor the source. The property of a scalar field as a function of space-time (physics) seems likely in harmony with the property of space-time (geometry). A space-time could be locally anisotropic but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

In the case of time-independent, the relation between the phase and the refractive index (8) implies that the phase transformation reduces to the refractive index transformation (9) where the transformation is deter-mined solely by the value of the distance from the origin or the sources. If we substitute eq.(12) into eq.(9), we obtain the relation below

$$q(\vec{r}) = X \int_{r_1}^{r_2} n_0 \left(1 - \frac{ar^2}{2}\right) d^3r \quad . \tag{13}$$

It means that as the distance decreases (increases) then the values of the refractive index and the phase increase (decrease). The refractive index is a number and the phase is an angle (in radians or degrees). Probably, we could interpret the refractive index as a winding number (a topological invariant) if the refractive index is an integer number.

Referring to our numerical result for calculating the weak field, a maxima peak point of the amplitude ρ = 0.55853 is related to magnitude of the magnetic field $|\vec{B}|$ = 0.10452. This very small but non-zero value of the magnetic field has a deep consequence related to the topological field (Chern-Simons) theory where the Chern-Simons action could be related topological object (helicity or knot). Thus, a weak subset field could result in the existence of the geometric optical helicity or knot [23,26]. This will be discussed in a separate article.

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