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Electromagnetic Wave Equation on Differential Form Representation

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ABSTRACT

One of the indispensable part of the theoretical physics interest is geometry differential. This one interest of physical area has been developed such as in electromagnetism. Maxwell's equations have been generalized in two covariant forms in differential form representation. A beautiful calculus vector in this representation, such as exterior derivative and Hodge star operator, lead this study. Electromagnetic wave equation has been expressed in differential form representation using Laplace-de Rham operator. Explicitly, wave equation shows the same form in Minkowski space-time like vector representation. This study is able to introduce us to learn application of differential form in physics.

Introduction

Electromagnetism as a basic science has a role to develop mathematics models. Earlier, In 1865 Maxwell proposed four equations called Maxwell's equations. These equations are the core or the summary of natural phenomenon that is related to electromagnetic fields. The form of the Maxwell equation in covariant form has been studied in many books (Griffith, 1999 and Jackson, 1999). Correspondingly, the study of physical phenomena in any coordinate system developed quite rapidly through the theory of mathematical geometry in a more general form. We start with the formulation of electromagnetic equations in differential form.

Electromagnetic studies in differential form representations have been carried out in various studies, which are written in various papers and texts such as [3-6]. In 1981 the relationship of external derivatives to electromagnetic equations in 4-dimensional space (space-time) was studied [3]. The relationship diagram of the electromagnetic equation is made in such a way that the exterior derivative successfully satisfies the electromagnetic equation. Electromagnetic studies in differential form representations focusing on visual advantages obtained from differential form representations have also been described [4]. The form order for electromagnetic quantities are presented in visual form for the more easily understood of electromagnetism [4]. The steps to reduce Maxwell's equations in differential form representations are well explained by Owere [5] and Hossine and Ali [6].

The study of electromagnetic equations in differential form representations which develop very rapidly allows electromagnetics to be studied in free coordinate space and other advantages. This encourages all matters related to electromagnetics to be assessed in the differential form representation. Actually, vector analysis is a real form of differential form, so differential form does not replace vectors. In particular cases, differential form and vector replace each other [4]. Therefore, in this paper, the explicitly component uses alternating differential forms and vectors.

Metric and Minkowski Space Field Tensor

The study of geometry and topology theory in physics has been written in many books. To understand many things about the application of geometrical concepts, especially the differential form can be seen in Nakahara [7]. However for more details, we start from the concept of tensor metric. A metric tensor introduces the length of a vector and an angle between every two vectors. The components of the metric are defined by the values of the scalar products of the basis vectors.

In elementary geometry, the inner product between two vectors **U** and **V** is defined by $\mathbf{U}\cdot\mathbf{V}=\sum_{i=1}^{m}U_iV_i$, where U_i and V_i are the components of the vectors in $\mathbb{R}^m$. On a manifold, an inner product is defined at each tangent space T_pM. Let M be a differentiable manifold. A Riemannian metric g on M is a type (0, 2) tensor field on M which satisfies the following axioms at each point $p \in M$ [7]:

(i) $g_p(U, V) = g_p(V,U)$,
(ii) $g_p(U,U) \geq 0$, where the equality holds only when $U = 0$.

A tensor field g of type (0, 2) is a pseudo-Riemannian metric if it satisfies
(i) and
(ii') if $g_p(U, V) = 0$ for any $U \in T_pM$, then $V = 0$.

If g is Riemannian, all the eigenvalues are strictly positive and if g is pseudo-Riemannian, some of them may be negative. If there are i positive and j negative eigenvalues, the pair (i,j) is called the index of the metric. If j = 1, the metric is called a Lorentz metric. Once a metric is diagonalized by an appropriate orthogonal matrix, it is easy to reduce all the diagonal elements to ±1 by a suitable scaling of the basis vectors with positive numbers. If we start with a Riemannian metric we end up with the Euclidean metric δ = diag(1, . . . , 1) and if we start with a Lorentz metric, the Minkowski metric η = diag(−1, 1, . . . , 1) [7].

Minkowski metric is the Lorentz metric on $\mathbb{R}^4$ that is written in terms of coordinates (ct; x; y; z) as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

where x, y and z are spatial dimensions, t is time dimension and c is speed of light. We use indices for space-time coordinates as follows:

$$ds^2 = -c^2\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{2}$$

which can be written as

$$ds^2 = \eta_{\mu\upsilon} dx^{\mu} dx^{\upsilon} \tag{3}$$

where $\eta_{\mu\upsilon}$ is the matrix

$$\eta_{\mu\upsilon} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

as we say before. This metric can be used to raising or lowering tensor index [7].

Maxwell Equation

Electromagnetism is lead by Maxwell's equations, which gives a calculation of how the non-permanent electric field can generate an impermanent magnetic field and vice versa. The four Maxwell equations are as follows.

$$\nabla \cdot \boldsymbol{D} = \rho \tag{5}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{7}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \tag{8}$$

In the equation above, **E** is an electric field, **B** is a magnetic flux density, **D** is a electric flux density, **H** is a magnetic field, ρ is a charge density (charge per unit volume), and **J** is the total current density. In addition, there is also a constant μ which is the medium permeability and ε is the permivisity of the medium. For vacuum space, then the electromagnetic entity has a reference value of μ_0 and ε_0. This value is a universal constant through a relationship that is c = 1= $1/\sqrt{\mu_0 \varepsilon_0}$, where c is the speed of light in a vacuum. The charge density and current density have a relationship as follow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{9}$$

called continuity equation.

Using vector calculus of divergence, from (6) we can obtain hat **B** must be a curl of a vector function, namely the vector potential A, can be write as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{10}$$

Substituting (10) into equation (7), we obtain

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) = 0$$

which means that the term in the bracket can be written as the gradient of a scalar function, namely the scalar potential ϕ

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{11}$$

Manifold and Differential Form

Let M is manifolds. Suppose that a tangent space on open subset $U \subset M$ as a vector space which is written by $T_x U$. A map $\omega_x = T_x U \to \mathbb{R}$ is a linier mapping from $T_x U$ to $\mathbb{R}$. Vector space consisting of all linear mapping is called the dual tangent space for $T_x U$ at x, denoted by $T_x^* U$.

Suppose there is a mapping $\omega_x = T_x U \times \cdots \times T_x U \to \mathbb{R}$ which is a multi linier mapping from the n tangent space $T_x U$ to $\mathbb{R}$. The vector space consisting of all multi linier mapping is referred to as space n-forms, denoted as $\Lambda^n(T_x U)$. In addition, there is a multi linier mapping $\omega_x^* = T_x^* U \times \cdots \times T_x^* U \to \mathbb{R}$ which is a multi linier mapping from n dual tangent space $T_x^* U$ to $\mathbb{R}$. The set of all mappings from the n dual tangent space is denoted by $\Lambda^n(T_x^* U)$.

Suppose $\beta_x : T_x^* U \times \cdots \times T_x^* U \times T_x U \times \cdots \times T_x U \to \mathbb{R}$ which is multi linier mapping from the q dual tangent spaces $T_x^* U$ and r tangent spaces $T_x U$ to $\mathbb{R}$. This multi linier map called as tensor type (q, r). Vector field can be defined as "a way of embedding" vector at a point. At $x \in U \subset M$, there are various ways of embedding that produces the tangent vector at the point x. Vector field X on $U \subset M$ can be written as a mapping

$$X : x \mapsto X(x).$$

Tensor fields have the same as the definition of a vector field. Tensor field of type (q, r) is a way of embedding the point $x$that produces a tensor of type (q, r). Tensor field β on U is differentiable cross section which can be written as

$$\beta : U \subset M \to \otimes^q T_x^* U \otimes^r T_x U; x \mapsto \beta(x) \in \otimes^q T_x^* U \otimes^r T_x U.$$

The same type of tensor fields on U form a vector space over $\mathbb{R}$ and form a module over ring. The tangent bundle TU on subset U is a collection of all the tangent space at U, i.e.

$$TU = \bigcup_{x \in M} T_x U.$$

Also, it can be defined dual tangent bundle, i.e.

$$T^* U = \bigcup_{x \in U} T_x^* U.$$

Manifolds U as a place of tangent bundle TU is defined as the basic space. More generally, defined n outer fiber bundle on U, which is as follows.

$$\Lambda^n U \equiv \Lambda^n(T^* U) := \{(x, \omega_x) | x \in U, \omega_x \in \Lambda^n(T_x^* U)\}.$$

This outer bundle is a bundle which its fiber is n outer algebra of dual tangent space $T_x^*U, x \in U$. Outer fiber bundle on base U define as

$$\Lambda^n U = \bigcup_{x \in U} \Lambda^n \left(T_x^* U\right).$$

Differential forms on differentiable manifold (M) can be defined in two points of view, namely from the viewpoint of algebra and geometry standpoint. Viewpoint of algebraic define differential forms as anti-symmetric multi linier mapping$\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$. Viewpoint of differential geometry define differential forms as differentiable cross-sectional

$$\omega: M \to \Lambda^n M$$

$$x \mapsto (x, \omega_x),$$

with $\omega_x = \omega(x) \in \Lambda^n(T_x^* M), \pi \circ \omega = Id_M$. Element n-differential forms can be written as

$$\omega = \frac{1}{n!} \omega_{\mu_1 \mu_2 \cdots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}.$$

On differential forms, defined multiplication operator, i.e. the wedge product $(\wedge)$. This operator is a totally anti-symmetric tensor product, i.e.

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} = \sum_{P \in S_n} sgn(\sigma) dx^{\mu_{\sigma(1)}} \otimes dx^{\mu_{\sigma(2)}} \otimes \cdots \otimes dx^{\mu_{\sigma(n)}}.$$

Exterior Derivative and Hodge Star Operator

Exterior derivative d is mapping $\Lambda^n M \to \Lambda^{n+1} M$ which is work on n-differential forms which defined as

$$d\omega = \frac{1}{n!}\left(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \mu_2 \cdots \mu_n} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}\right).$$

This operator has properties as follow.

1. For constants c_1, and c_2 and forms ϕ_1 and ϕ_2
   $d(c_1\phi_1 + c_2\phi_2) = c_1 d\phi_1 + c_2 d\phi_2$
2. For a 0-form i.e., a function $\varphi = A(x_1, x_2, \ldots, x_n)$
   $d\varphi = dA = A_{x1}(x_1, x_2, \ldots, x_n)dx_1 + \ldots + A_{xn}(x_1, x_2, \ldots, x_n)dx_n$
3. For a n-form i.e., a function $\varphi = A(x_1, x_2, \ldots, x_n)dx_{r1} \wedge dx_{r2} \wedge \ldots \wedge dx_{rn}$
   $d\varphi = (dA) \wedge (dx_{r1} \wedge dx_{r2} \wedge \ldots \wedge dx_{rn})$

In addition, there is also a Hodge $\star$ operator, which is mapping from $\Lambda^n M \to \Lambda^{m-n} M$ which is working on basis vector $\Lambda^n M$ which defined as follows.

$$\star\left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}\right) = \frac{\sqrt{|g|}}{(m-n)!} \epsilon^{\mu_1 \mu_2 \cdots \mu_n}{}_{\nu_{n+1} \cdots \nu_m} dx^{\nu_{n+1}} \wedge \cdots \wedge dx^{\nu_m}.$$

where ϵ is the totally anti-symmetric Levi-Civita permutation symbol. How this operator work to the basis form have been written in Hossine [6].

Electromagnetic wave in Differential Form

Maxwell equations with the source of the charge or current source is generally written in the four field vectors in 3-dimensional space, namely the electric field $\boldsymbol{E}$ magnetic field $\boldsymbol{H}$, electric flux density $\boldsymbol{D}$, and magnetic flux density $\boldsymbol{B}$. The electric field is replaced in the form of 1-differential form E and magnetic flux density replaced by 2-differential forms B, which can be written as

$$E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3 \quad (12)$$

$$B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \quad (13)$$

In space-time, we consider potential A as 1-form which component is A^0, A^1, A^2, A^3. In vector representation, $A^0 = \phi$ and A^1, A^2, A^3 are the component of potential vector $\boldsymbol{A}$. Taking the exterior derivative to the A, and we remember the relation of (10) and (11), we obtain

$$\begin{aligned} dA = {} & E_1 dx^1 \wedge dx^0 + E_2 dx^2 \wedge dx^0 + E_3 dx^3 \wedge dx^0 + \\ & B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \end{aligned}$$

This equation is called electromagnetic field F that is

$$F = dA = B + E \wedge dx^0 \quad (14)$$

Taking again exterior derivative to the (14), we obtain

$$\begin{aligned} dF = {} & (\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3) dx^1 \wedge dx^2 \wedge dx^3 + (\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1) dx^0 \wedge dx^2 \wedge dx^3 + \\ & (\partial_3 E_1 - \partial_1 E_3 + \partial_0 B_2) dx^0 \wedge dx^3 \wedge dx^1 + (\partial_1 E_2 - \partial_2 E_1 + \partial_0 B_3) dx^0 \wedge dx^1 \wedge dx^2 \end{aligned}$$

From the properties of exterior derivative, we obtain that $dF = d^2 A = 0$, then the result the same as

$$\begin{aligned} \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 &= 0 \\ \partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 &= 0 \\ \partial_3 E_1 - \partial_1 E_3 + \partial_0 B_2 &= 0 \\ \partial_1 E_2 - \partial_2 E_1 + \partial_0 B_3 &= 0 \end{aligned}$$

These equations have the same pattern with (6) and (7). So, $dF = 0$ called Homogenous Maxwell's equations in differential form representation. Starting form (13) and taking Hodge star operator we obtain

$$\begin{aligned} \star\, dA = {} & -B_1 dx^1 \wedge dx^0 - B_2 dx^2 \wedge dx^0 - B_3 dx^3 \wedge dx^0 + \\ & E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2 \end{aligned} \quad (15)$$

Before next step, we define 1-form charge density and current density, namely J. So in space-time, J has component $J_1 dx^1 + J_2 dx^2 + J_3 dx^3 - \rho dx^0$. It is clear that 1-form J is combination of charge density and current density in vector space. Back to the (15), taking exterior derivative and Hodge star operator to the (15), for the step as [6], we obtain

$$\star d \star F = -(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3)dx^0 + (\partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1)dx^1 \\ +(\partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2)dx^2 + (\partial_2 B_1 - \partial_1 B_2 - \partial_0 E_3)dx^3 \quad (16)$$

We set $\star d \star F = J$, then the same as

$$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \rho$$
$$\partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 = J^1$$
$$\partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2 = J^2$$
$$\partial_2 B_1 - \partial_1 B_2 - \partial_0 E_3 = J^3$$

These equations have the same pattern with (5) and (8). So, $\star d \star F = J$ called Inhomogeneous Maxwell's equations in differential form representation. Now we get two covariant Maxwell's equation in differential form. It surprisingly that the differential forms of Maxwell's equations are exactly the same as the covariant forms when expressed in terms of components.

Next, we will show that how the calculus in differential form have the same form as it component on wave equation. For that, first we define Laplace operator in differential form representation. Unfortunately there is not a simple case, because if taking exterior derivative twice to the differential form, there will be vanish ($d^2\alpha = 0$).

On differential form, Laplace operator define by the Laplace-de Rham operator [8]

$$\Delta\alpha = (-1)^{n(p+1)}[(-1)^n \delta d + d\delta]\alpha \quad (17)$$

where $\delta = \star d \star$. Let we check for the electric wave equation in Minkowski space-time. In vector representation, this equation is $\Delta E = \mu\varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial J}{\partial t}$. We know that E is 1-form, so do J. So, on right hand, there are 1-form, as the derivative with time did not change form order. We take $n = 4$ and $p = 1$ on (17), we obtain

$$\Delta\alpha = [\delta d + d\delta]\alpha \quad (18)$$

so in left hand, following the properties of exterior derivative and Hodge star operator, the Laplace-de Rham did not change form order. This obvious way to check the equality. In simple way, the wave equation in differential form representation can be written as

$$[\delta d + d\delta]E = \mu\varepsilon\partial_0^2 E + \mu\partial_0 J \quad (19)$$

In component form, the calculation explicitly of Laplace-de Rham operator as follow. Taking hodge star operator to E we obtain

$$\star E = E_1(\star dx^1) + E_2(\star dx^2) + E_3(\star dx^3) \\ = -E_1 dx^0 \wedge dx^2 \wedge dx^3 + E_2 dx^0 \wedge dx^1 \wedge dx^3 - E_3 dx^0 \wedge dx^1 \wedge dx^2$$

Taking exterior derivative we obtain

$$d\star E=\left(\frac{\partial E_1}{\partial x^1}+\frac{\partial E_2}{\partial x^2}+\frac{\partial E_3}{\partial x^3}\right)dx^0\wedge dx^1\wedge dx^2\wedge dx^3$$

Taking hodge star operator we obtain

$$\begin{aligned}\star d\star E&=\left(\frac{\partial E_1}{\partial x^1}+\frac{\partial E_2}{\partial x^2}+\frac{\partial E_3}{\partial x^3}\right)\star\left(dx^0\wedge dx^1\wedge dx^2\wedge dx^3\right)\\&=\left(\frac{\partial E_1}{\partial x^1}+\frac{\partial E_2}{\partial x^2}+\frac{\partial E_3}{\partial x^3}\right)(-1)\\&=-\left(\frac{\partial E_1}{\partial x^1}+\frac{\partial E_2}{\partial x^2}+\frac{\partial E_3}{\partial x^3}\right)\end{aligned}$$

Again, taking exterior derivative we obtain

$$\begin{aligned}d\star d\star E=&-\frac{\partial^2E_1}{\partial(x^1)^2}dx^1-\frac{\partial^2E_1}{\partial x^1\partial x^2}dx^2-\frac{\partial^2E_1}{\partial x^1\partial x^3}dx^3-\frac{\partial^2E_1}{\partial x^1\partial x^0}dx^0-\frac{\partial^2E_2}{\partial x^2\partial x^1}dx^1\\&-\frac{\partial^2E_2}{\partial(x^2)^2}dx^2-\frac{\partial^2E_2}{\partial x^2\partial x^3}dx^3-\frac{\partial^2E_2}{\partial x^2\partial x^0}dx^0-\frac{\partial^2E_3}{\partial x^3\partial x^1}dx^1-\frac{\partial^2E_3}{\partial x^3\partial x^2}dx^2\\&-\frac{\partial^2E_3}{\partial(x^3)^2}dx^3-\frac{\partial^2E_3}{\partial x^3\partial x^0}dx^0\end{aligned}$$

Then

$$\begin{aligned}d\star d\star E=&-\left(\frac{\partial^2E_1}{\partial x^1\partial x^0}+\frac{\partial^2E_2}{\partial x^2\partial x^0}+\frac{\partial^2E_3}{\partial x^3\partial x^0}\right)dx^0-\left(\frac{\partial^2E_1}{\partial(x^1)^2}+\frac{\partial^2E_2}{\partial x^2\partial x^1}+\frac{\partial^2E_3}{\partial x^3\partial x^1}\right)dx^1\\&-\left(\frac{\partial^2E_1}{\partial x^1\partial x^2}+\frac{\partial^2E_2}{\partial(x^2)^2}+\frac{\partial^2E_3}{\partial x^3\partial x^2}\right)dx^2-\left(\frac{\partial^2E_1}{\partial x^1\partial x^3}+\frac{\partial^2E_2}{\partial x^2\partial x^3}+\frac{\partial^2E_3}{\partial(x^3)^2}\right)dx^3\end{aligned}$$

Now, taking exterior derivative to E we obtain

$$\begin{aligned}dE=&\frac{\partial E_1}{\partial x^0}dx^0\wedge dx^1+\frac{\partial E_1}{\partial x^2}dx^2\wedge dx^1+\frac{\partial E_1}{\partial x^3}dx^3\wedge dx^1+\frac{\partial E_2}{\partial x^0}dx^0\wedge dx^2+\frac{\partial E_2}{\partial x^1}dx^1\\&\wedge dx^2+\frac{\partial E_2}{\partial x^3}dx^3\wedge dx^1+\frac{\partial E_3}{\partial x^0}dx^0\wedge dx^3+\frac{\partial E_3}{\partial x^1}dx^1\wedge dx^3+\frac{\partial E_3}{\partial x^2}dx^2\\&\wedge dx^3\end{aligned}$$

Taking the hodge star operator, we obtain

$$\begin{aligned}\star dE=&\left(\frac{\partial E_3}{\partial x^2}-\frac{\partial E_2}{\partial x^3}\right)dx^0\wedge dx^1+\left(\frac{\partial E_1}{\partial x^3}-\frac{\partial E_3}{\partial x^1}\right)dx^0\wedge dx^2+\left(\frac{\partial E_2}{\partial x^1}-\frac{\partial E_1}{\partial x^2}\right)dx^0\wedge dx^3+\frac{\partial E_2}{\partial x^0}dx^1\\&\wedge dx^3-\frac{\partial E_3}{\partial x^0}dx^1\wedge dx^2-\frac{\partial E_1}{\partial x^0}dx^2\wedge dx^3\end{aligned}$$

Taking the exterior derivative, we obtain

$$\begin{aligned} d\star dE = &\left(-\frac{\partial^2 E_1}{\partial x^3\partial x^1}-\frac{\partial^2 E_2}{\partial x^3\partial x^2}-\frac{\partial^2 E_3}{\partial (x^0)^2}+\frac{\partial^2 E_3}{\partial (x^1)^2}+\frac{\partial^2 E_3}{\partial (x^2)^2}\right)dx^0\wedge dx^1\wedge dx^2 \\ &+\left(\frac{\partial^2 E_1}{\partial x^2\partial x^1}+\frac{\partial^2 E_3}{\partial x^2\partial x^3}+\frac{\partial^2 E_2}{\partial (x^0)^2}-\frac{\partial^2 E_2}{\partial (x^1)^2}-\frac{\partial^2 E_2}{\partial (x^3)^2}\right)dx^0\wedge dx^1\wedge dx^3 \\ &+\left(-\frac{\partial^2 E_2}{\partial x^1\partial x^2}-\frac{\partial^2 E_3}{\partial x^1\partial x^3}-\frac{\partial^2 E_1}{\partial (x^0)^2}+\frac{\partial^2 E_1}{\partial (x^2)^2}+\frac{\partial^2 E_1}{\partial (x^3)^2}\right)dx^0\wedge dx^2\wedge dx^3 \\ &+\left(-\frac{\partial^2 E_1}{\partial x^0\partial x^1}-\frac{\partial^2 E_2}{\partial x^0\partial x^2}-\frac{\partial^2 E_3}{\partial x^0\partial x^3}\right)dx^1\wedge dx^2\wedge dx^3 \end{aligned}$$

Taking Hodge star again, we get

$$\begin{aligned} \star d\star dE = &\left(\frac{\partial^2 E_1}{\partial x^3\partial x^1}+\frac{\partial^2 E_2}{\partial x^3\partial x^2}+\frac{\partial^2 E_3}{\partial (x^0)^2}-\frac{\partial^2 E_3}{\partial (x^1)^2}-\frac{\partial^2 E_3}{\partial (x^2)^2}\right)dx^3 \\ &+\left(\frac{\partial^2 E_1}{\partial x^2\partial x^1}+\frac{\partial^2 E_3}{\partial x^2\partial x^3}+\frac{\partial^2 E_2}{\partial (x^0)^2}-\frac{\partial^2 E_2}{\partial (x^1)^2}-\frac{\partial^2 E_2}{\partial (x^3)^2}\right)dx^2 \\ &+\left(\frac{\partial^2 E_2}{\partial x^1\partial x^2}+\frac{\partial^2 E_3}{\partial x^1\partial x^3}+\frac{\partial^2 E_1}{\partial (x^0)^2}-\frac{\partial^2 E_1}{\partial (x^2)^2}+\frac{\partial^2 E_1}{\partial (x^3)^2}\right)dx^1 \\ &+\left(\frac{\partial^2 E_1}{\partial x^0\partial x^1}+\frac{\partial^2 E_2}{\partial x^0\partial x^2}+\frac{\partial^2 E_3}{\partial x^0\partial x^3}\right)dx^0 \end{aligned}$$

Adding $(\star d\star dE + d\star d\star E) = [\delta d + d\delta]E$, we get

$$\begin{aligned} [\delta d + d\delta]E = &\frac{\partial^2 E_1}{\partial (x^0)^2}dx^1+\frac{\partial^2 E_2}{\partial (x^0)^2}dx^2+\frac{\partial^2 E_3}{\partial (x^0)^2}dx^3-\frac{\partial^2 E_1}{\partial (x^1)^2}dx^1-\frac{\partial^2 E_1}{\partial (x^2)^2}dx^1 \\ &-\frac{\partial^2 E_1}{\partial (x^3)^2}dx^1-\frac{\partial^2 E_2}{\partial (x^1)^2}dx^2-\frac{\partial^2 E_2}{\partial (x^2)^2}dx^2-\frac{\partial^2 E_2}{\partial (x^3)^2}dx^2-\frac{\partial^2 E_3}{\partial (x^1)^2}dx^3 \\ &-\frac{\partial^2 E_3}{\partial (x^2)^2}dx^3-\frac{\partial^2 E_3}{\partial (x^3)^2}dx^3 \\ = &\frac{\partial^2 E}{\partial (x^0)^2}-\frac{\partial^2 E}{\partial (x^1)^2}-\frac{\partial^2 E}{\partial (x^2)^2}-\frac{\partial^2 E}{\partial (x^3)^2} \end{aligned}$$

This is laplacian in Minkowski space-time. Additional term derivative $\frac{\partial^2 E}{\partial (x^0)^2}$ shows calculus in differential form following vector representation in Minkowski space-time. The same way can applied to the 2-form magnetic field B wave equation.

Conclusion

In this paper, we studied the electromagnetic theory in $\mathbb{R}^4$ spacetime. By using differential forms, we formulate electromagnetic wave on differential form representation. This studied show that when expressed in terms of component, differential form can back again like equation in vector form. But, we know that differential form more general than vector as we desire.

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